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Sharp existence results for mean field equations with singular data

D. Bartolucci [a,*], C.S. Lin [b]

[a] Department of Mathematics, University of Rome "Tor Vergata", Via della Ricerca Scientifica n.1, 00133 Rome, Italy
[b] Taida Institute for Mathematical Sciences and Department of Mathematics, National Taiwan University, Taipei, Taiwan

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ABSTRACT

Let Ω be a simply connected, open and bounded domain in $\mathbb{R}^2$. We are concerned with the nonlinear elliptic problem

$$\begin{cases} -\Delta v = 8\pi \dfrac{e^v}{\int_\Omega e^v} - 4\pi \displaystyle\sum_{j=1}^m \alpha_j \delta_{p_j} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where $\alpha_j > 0$, δ_{p_j} denotes the Dirac mass with singular point p_j and $\{p_1, \ldots, p_m\} \subset \Omega$. We provide necessary and sufficient conditions for the existence of solutions to (0.1). Our result is the two dimensional version of the sharp existence/nonexistence result obtained in Druet (2002) [13] for elliptic equations with critical exponent in dimension 3. In particular, we prove that the set $\Omega_+^m(\underline{\alpha})$ is open, where, for a given $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \subset (0, +\infty) \times \cdots \times (0, +\infty)$, $\Omega_+^m(\underline{\alpha}) = \{(p_1, \ldots, p_m) \mid$ problem (0.1) has a solution$\} \subset \Omega \times \cdots \times \Omega$.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a simply connected, open and bounded domain and $\{p_1, \ldots, p_m\} \subset \Omega$ be any finite subset. We are concerned with the existence of solutions for

* Corresponding author.
E-mail addresses: bartoluc@mat.uniroma2.it (D. Bartolucci), cslin@math.ntu.edu.tw (C.S. Lin).

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$$\begin{cases} -\Delta v = \lambda \dfrac{e^v}{\int_\Omega e^v} - 4\pi \displaystyle\sum_{j=1}^m \alpha_j \delta_{p_j} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in case $\lambda = 8\pi$ and $\alpha_j \in (0, +\infty)$, $\forall j \in \{1, \dots, m\}$.

The analysis of (1.1) has recently attracted a lot of attention due its many applications in mathematical physics. We refer the reader to [2,4,5,7–12,14,15,17,19–22], and the references quoted therein for further details. In particular we refer to [7,8,16] and the introduction of [4] for the application of (1.1) to the analysis of vortex-type configurations in turbulent Euler flows.

We will not discuss here issues related with non-smooth domains. Therefore, unless otherwise specified we will assume that Ω is of class C^2.

We will denote by $z = x_1 + ix_2$ and $p_j = p_{j,1} + ip_{j,2}$ the complex coordinates corresponding to $(x_1, x_2) \in \Omega$, $(p_{j,1}, p_{j,2}) \in \Omega$, and by $D = \{z \in \mathbb{R}^2 \mid |z| < 1\}$ the open unit disk.

We will often need to use conformal mappings from D to Ω. To avoid any possible ambiguity, in case Ω itself is the unit disk, we will denote it by B_1. In particular, for any fixed $p \in \Omega$, we will denote by $f_p : D \mapsto \Omega$, any Riemann map which satisfies $f_p(0) = p$, and set $g_p = f_p^{-1} : \Omega \mapsto D$ to be its inverse.

For any fixed $\alpha > 0$, $p \in \Omega$ and any $r > 0$ small enough, let $G_\Omega(z, p) \in C^0(\overline{\Omega} \setminus B_r(p))$ be the unique solution of

$$\begin{cases} -\Delta G_\Omega(z, p) = \delta_p & \text{in } \Omega, \\ G_\Omega(z, p) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The continuity assumption ensures that G_Ω is uniquely defined by $G_\Omega(z, p) = -\frac{1}{2\pi} \log |g_p(z)|$.

Let $\tilde{G}_\Omega$ denote the regular part of $G_\Omega(z, p)$, and set $z = f_p(w)$. Then, we have

$$\tilde{G}_\Omega(z, p) = G_\Omega(z, p) + \frac{1}{2\pi} \log |z - p| = -\frac{1}{2\pi} \log \frac{|g_p(z)|}{|z - p|} = \frac{1}{2\pi} \log \frac{|f_p(w) - f_p(0)|}{|w|},$$

and we define

$$\gamma_\Omega(z) = \tilde{G}_\Omega(z, z) = \frac{1}{2\pi} \log \frac{(1 - |g_p(z)|^2)}{|g_p'(z)|} = \frac{1}{2\pi} \log\big(1 - |w|^2\big)\big|f_p'(w)\big|$$

to be the corresponding Robin function.

We define v to be a solution for (1.1) if $u := v + 4\pi \sum_{j=1,\dots,m} \alpha_j G_\Omega(z, p_j)$ is an $H_0^1(\Omega)$, weak solution for

$$\begin{cases} -\Delta u = \lambda \dfrac{V e^u}{\int_\Omega V e^u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where

$$V(z) = \exp\Bigg(-4\pi \sum_{j=1,\dots,m} \alpha_j G_\Omega(z, p_j)\Bigg). \tag{1.4}$$

Two crucial results (Theorems 1.1 and 1.4 below) will be used whose proofs will not be discussed here. We refer the reader to the remarks following the corresponding statements for further details.

Theorem 1.1. *(See [3].) Let* $\log h$ *be harmonic and continuous in* $\overline{\Omega}$. *For any* $\lambda \in (0, 8\pi)$ *there exists one and only one solution* u_λ *for*

$$\begin{cases} -\Delta u = \lambda \dfrac{hVe^u}{\int_\Omega hVe^u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

In particular (1.5) *admits at most one solution for* $\lambda = 8\pi$. *Moreover, for any* $\lambda \in (0, 8\pi]$ *the first eigenvalue of the linearized problem for* (1.5) *at* u_λ *is strictly positive.*

Remark. Theorem 1.1 was proved in [3] in case $h \equiv 1$. It is straightforward to verify that the same proof works for any h satisfying the required assumptions as well. We omit the details of this proof here since it can be worked out by a step-by-step adaptation of the one already provided in [3].

Remark 1. We remark that uniqueness for (1.5) holds for example if Ω admits a finite number of conical or cuspidal (non-exponential) points, see [9] and [3] for more details.

Clearly u solves (1.3) if and only if it is a critical point for

$$J_\lambda(u) = -\frac{1}{2\lambda}\int\limits_\Omega |\nabla u|^2 + \log \fint\limits_\Omega Ve^u, \quad u \in H_0^1(\Omega), \tag{1.6}$$

where $\fint_\Omega = \frac{1}{|\Omega|}\int_\Omega$. As a consequence of the Moser–Trudinger inequality [18] we see that J_λ attains its maximum for any $\lambda \in (0, 8\pi)$. Indeed, J_λ is bounded from above, upper-semicontinuous and coercive for any $\lambda \in (0, 8\pi)$. A subtle problem arises for $\lambda = 8\pi$, since $J_{8\pi}$ is bounded from above but not coercive.

Then the existence of solutions for (1.3), or equivalently of maximizers for $J_{8\pi}$ is not granted. In the case where no singularities are contained in Ω, that is $\alpha_j = 0$ $\forall j \in \{1, \dots, m\}$, this problem has been solved in [9].

However the rest of our discussion is more delicate than that in [9] because of the Dirac masses in (1.1). This issue affects the proofs of our main results in various ways. For example in [9] some results obtained in [8] were used which cannot be taken for grant in our situation. As a matter of fact, even the definitions of the main quantities involved in our analysis have to be carefully studied to take into account the role of the Dirac masses.

Therefore we provide an ab initio and self-contained discussion of the problem which generalizes that of the "regular" case $\alpha_j = 0$ $\forall j \in \{1, \dots, m\}$.

Let us define

$$I_\lambda(\Omega) = \sup_{u \in H_0^1(\Omega)} J_\lambda(u), \tag{1.7}$$

and

$$F_m(z; q, \Omega) = 8\pi \tilde{G}_\Omega(z, q) + \log\big[V(z)\big], \tag{1.8}$$

$$\tilde{F}_m(z; \Omega) = 4\pi \gamma_\Omega(z) + \log\big[V(z)\big]. \tag{1.9}$$

Our first result toward the understanding of the existence/nonexistence problem for $\lambda = 8\pi$ is a generalization of the one obtained in [8] in the "regular" case. Indeed, we have the following

Theorem 1.2.

$$I_{8\pi}(\Omega) \geqslant 1 + \max_{\overline{\Omega}} \tilde{F}_m(\cdot;\Omega) + \log\frac{|B_1|}{|\Omega|}, \tag{1.10}$$

and if the strict inequality holds in (1.10), *then* $I_{8\pi}(\Omega)$ *is attained.*

Next, as a consequence of Theorem 1.1, we prove the following

Theorem 1.3. *Let* $\{u_\lambda\}$ *be the family of maximizers for* (1.3) *for* $\lambda \in (0, 8\pi)$. *The following properties are equivalent*:

(i) *There exists* $C > 0$ *such that* $\sup_{\lambda\in(0,8\pi)} \|u_\lambda\|_\infty \leqslant C$;
(ii) $I_{8\pi}$ *is attained*;
(iii) *Problem* (1.3) *admits a solution for* $\lambda = 8\pi$.

Here and in the rest of this paper we will denote by

$$d\tau(z) = \frac{|dz \wedge d\bar{z}|}{2}, \qquad d\tau(w) = \frac{|dw \wedge d\overline{w}|}{2}$$

the volume element corresponding to the coordinates $z \in \Omega$ and $w \in D$ respectively.
Then we have

Theorem 1.4. *Let* $\log h$ *be harmonic and continuous in* $\overline{\Omega}$. *Assume that* $\{u_k\}$ *is a one-point blow-up sequence for*

$$\begin{cases} -\Delta u = \lambda \dfrac{hVe^u}{\int_\Omega hVe^u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

that is, as $k \to +\infty$, *suppose that* $\{u_k\}$ *satisfies*

$$\lambda_k \to 8\pi, \qquad \lambda_k \frac{hVe^{u_k}}{\int_\Omega hVe^{u_k}} \rightharpoonup 8\pi\delta_q, \tag{1.11}$$

weakly in the sense of measures in Ω, *for some* $q \in \Omega$. *Put*

$$\varepsilon_k := \lambda_k \left(\int_\Omega hVe^{u_k} \right)^{-1}. \tag{1.12}$$

Then, as $k \to +\infty$, *we have* $\varepsilon_k \to 0^+$ *and*

$$\lambda_k - 8\pi = h(q)V(q)\varepsilon_k \left(\int_\Omega \frac{H(z,q)}{|z-q|^4}\, d\tau(z) - \int_{\mathbb{R}^2\setminus\Omega} \frac{1}{|z-q|^4}\, d\tau(z) + o(1) \right), \tag{1.13}$$

where

$$H(z,q) = \frac{h(z)V(z)}{h(q)V(q)} \exp\bigl(8\pi \tilde{G}_\Omega(z,q) - 8\pi\gamma(q)\bigr) - 1. \tag{1.14}$$

Remark 2. It is well known that q, the blow-up point, must be a critical point for $F_m(\cdot;q,\Omega)+\log[h]$ or equivalently, a critical point for $\tilde{F}_m(\cdot;\Omega)+\log[h]$. In particular, by using (1.11) and the mass-quantization result in [5], we see that $q\notin\{p_1,\dots,p_m\}$.

Remark. We will not provide the proof of Theorem 1.4 here since it can be worked out by a step-by-step adaptation of the one already worked out in [9]. Indeed, one has just to use Remark 2 and the fact that $\log(h(z)V(z))$ is harmonic in $\Omega\setminus\{p_1,\dots,p_m\}$. We refer the reader to [9] for more details concerning this point.

Let q be any critical point for $F_m(\cdot;q,\Omega)$. For each $j=1,\dots,m$, we choose $p_j^*\in D$ to satisfy

$$f_q\big(p_j^*\big)=p_j. \tag{1.15}$$

Putting $z=f_q(w)$ and $f_j=f_{p_j}$, we have

$$V\big(f_q(w)\big)=\exp\Bigg(-4\pi\sum_{j=1,\dots,m}\alpha_j G_\Omega\big(f_q(w),f_q\big(p_j^*\big)\big)\Bigg)=\prod_{j=1,\dots,m}\big|f_j^{-1}\big(f_q(w)\big)\big|^{2\alpha_j}.$$

It is well known that $\phi:D\mapsto D$ is univalent if and only if it takes the form

$$\phi_{\sigma,\theta}(w)=e^{i\theta}\frac{\sigma-w}{1-\overline{\sigma}w},$$

for some $\sigma\in D$, and $\theta\in[0,2\pi)$. Moreover, observe that $f_q:D\mapsto\Omega$, $f_j:D\mapsto\Omega$, $f_q(p_j^*)=p_j$, $f_j(0)=p_j$. Then,

$$f_j^{-1}\big(f_q(w)\big)=\phi_{p_j^*,\theta_j}(w)=e^{i\theta_j}\frac{p_j^*-w}{1-\overline{p_j^*}w},$$

for some $\theta_j\in[0,2\pi)$, and

$$V\big(f_q(w)\big)=\prod_{j=1,\dots,m}\big|f_j^{-1}\big(f_q(w)\big)\big|^{2\alpha_j}=\prod_{j=1,\dots,m}\left|\frac{p_j^*-w}{1-\overline{p_j^*}w}\right|^{2\alpha_j}, \tag{1.16}$$

where the values $\{p_1^*,\dots,p_m^*\}$ are determined implicitly by (1.15).

We also define

$$r_1=\min_{j\in\{1,\dots,m\}}\big|p_j^*\big| \tag{1.17}$$

to be the minimal radius attained by $\{p_1^*,\dots,p_m^*\}$. We observe that, since $f_q(0)=q$, $q\notin\{p_1,\dots,p_m\}$ and f_q is univalent, then

$$p_j^*\neq 0,\quad \forall j\in\{1,\dots,m\},\quad \text{and then}\quad r_1>0. \tag{1.18}$$

Set

$$\Phi(w)=\prod_{j=1,\dots,m}\left(\frac{p_j^*-w}{1-\overline{p_j^*}w}\right)^{\alpha_j}, \tag{1.19}$$

and $B_{r_1} = B_{r_1}(0)$. Then, let

$$\Phi(w) f_q'(w) = \sum_{k=0}^{+\infty} c_k w^k, \quad w \in B_{r_1},$$

be the power series expansion for $\Phi(w) f_q'(w)$ in B_{r_1}. Since $q \in \Omega$ is a critical point for $F_m(\cdot; q, \Omega)$, then $w = 0 \in D$ is a critical point in the transplanted domain D for $F_m(\cdot; 0, D)$. Indeed, we observe that in the transplanted domain D, the weight function (see also (3.1) and (3.5) below) corresponding to problem (1.3) reads $V(f(w))|f_q'(w)|^2$. In particular, since $\tilde{G}_D(w, 0) \equiv 0$, we have

$$\left|\Phi(w) f_q'(w)\right|^2 = \exp\bigl(F_m(w; 0, D)\bigr),$$

and then $w = 0$ must be a critical point for $\Phi(w) f_q'(w)$. We conclude that $c_1 = 0$, so that we can define:

$$S(q) = -\frac{1}{r_1^2}|c_0|^2 + \sum_{k=2}^{+\infty} \frac{|c_k|^2}{k-1} r_1^{2(k-1)} + \frac{1}{\pi} \int\limits_{D \setminus B_{r_1}} \frac{|\Phi(w) f_q'(w)|^2}{|w|^4}\, d\tau(w). \tag{1.20}$$

Clearly, since f_q is univalent, and in view of (1.18), we have

$$c_0 = \left(\prod_{j=1,\dots,m} \left|p_j^*\right|^{2\alpha_j}\right) \left|f_q'(0)\right|^2 \neq 0. \tag{1.21}$$

By using Theorem 1.4, we will prove

Theorem 1.5. *Let $\{u_k\}$ be a one-point blow-up sequence for* (1.3)*, which then satisfies, as $k \to +\infty$,*

$$\lambda_k \to 8\pi, \qquad \lambda_k \frac{V e^{u_k}}{\int_\Omega V e^{u_k}} \rightharpoonup 8\pi\,\delta_q, \tag{1.22}$$

weakly in the sense of measures in Ω, for some $q \in \Omega$. Put

$$\sigma_k := \lambda_k \left(\int\limits_\Omega V e^{u_k}\right)^{-1}.$$

Then, as $k \to +\infty$, we have $\sigma_k \to 0^+$ and

$$\lambda_k - 8\pi = \pi \sigma_k \bigl(S(q) + o(1)\bigr), \tag{1.23}$$

with S defined by (1.20)*.*

By using Theorem 1.1 and Theorems 1.4, 1.5, we will prove the main result of this paper. In particular our result yields the two dimensional version of the sharp existence/nonexistence result obtained in [13] for elliptic equations with critical exponent in dimension 3. Indeed, we have the following

Theorem 1.6. *Let q be a relative maximizer for $\tilde{F}_m(\cdot; \Omega)$, with $S(q) \leqslant 0$. Then q is the unique absolute maximizer of $\tilde{F}_m(\cdot; \Omega)$ and in particular the set of maximizers $\{u_\lambda\}$ for J_λ with $\lambda \in (0, 8\pi)$ satisfies* (1.22) *with concentration point q.*

Remark. By using the notion of stability of critical points for $\tilde{F}_m(\cdot;\Omega)$ introduced in [14], we can prove that if q is *any* critical point for $\tilde{F}_m(\cdot;\Omega)$, with $S(q)\leqslant 0$, then q is the unique absolute maximizer and the conclusion of Theorem 1.6 holds true, in this case as well. The proof can be worked out by the same argument used to prove Theorem 1.6. We skip it here to avoid technicalities. We remark that this result has been firstly obtained in [9] in case $\alpha_j=0$ $\forall j\in\{1,\dots,m\}$.

As an immediate consequence of Theorem 1.6, we have

Corollary 1.7. *Let $\{u_k\}$ be a sequence of solutions which satisfies* (1.11)*. If $|\lambda_k-8\pi|=o(1)\varepsilon_k$ as $k\to+\infty$, where ε_k has been defined in* (1.12)*, then $\lambda_k<8\pi$.*

Remark. Corollary 1.7 is *false* if Ω is a torus, see [12].

For a given $\underline{\alpha}=(\alpha_1,\dots,\alpha_m)\subset(0,+\infty)\times\dots\times(0,+\infty)$, let us define

$$\Omega_+^m(\underline{\alpha})=\big\{(p_1,\dots,p_m)\ \big|\ \text{problem (0.1) has a solution}\big\}\subset\Omega\times\dots\times\Omega.$$

Then, another immediate consequence of Theorem 1.6 is the following,

Corollary 1.8. *The set $\Omega_+^m(\underline{\alpha})$ is open.*

A subtle and interesting problem to solve is that of the topology of $\Omega_+^m(\underline{\alpha})$. We will investigate this problem in another paper. Meanwhile, we have the following

Conjecture 1. *If Ω is convex then $\Omega_+^m(\underline{\alpha})$ is simply connected.*

Conjecture 2. *If $\Omega=B_1$, $p_1=0$, $\alpha_2=\dots=\alpha_m$ and $\{p_2,\dots,p_m\}$ is symmetric, then* (0.1) *has a solution.*

As a matter of fact, the analysis of the sign of $S(q)$ as a function of $\{\alpha_j\}$ and $\{p_j\}$, $j\in\{1,\dots,m\}$, is a delicate problem on its own. Even in the case where Ω take up very simple geometries, a complete characterization of the existence/nonexistence problem for (0.1) is nontrivial. We illustrate this fact in Sections 5, 6, 7 by discussing some explicit examples, where we will use various consequences of Theorem 1.6. Indeed, we have

Theorem 1.9. *$I_{8\pi}(\Omega)$ is attained if and only if $S(q)>0$ for an absolute maximizer q of $\tilde{F}_m(\cdot;\Omega)$.*

As a consequence of Theorems 1.9 and 1.1, we also have

Theorem 1.10. *$I_{8\pi}(\Omega)$ is attained if and only if $I_{8\pi}>1+\max_{\overline{\Omega}}\tilde{F}_m(\cdot;\Omega)+\log\frac{|B_1|}{|\Omega|}$.*

This result is the two dimensional version of the sharp existence/nonexistence result obtained in [13] for elliptic equations with critical exponent in dimension 3.

Moreover, as an immediate consequence of Theorems 1.6 and 1.9, we have

Corollary 1.11. *If $\tilde{F}_m(\cdot;\Omega)$ admits more than one absolute maximizer, then $I_{8\pi}(\Omega)$ is attained.*

This paper is organized as follows. In Section 2 we will prove Theorem 1.2 and some results which will be used in the proofs of Theorems 1.6, 1.9, 1.10. In Section 3 we will prove Theorems 1.5 and 1.6. In Section 4 we will prove Theorems 1.9, 1.10 and Corollary 1.8. In Sections 5, 6 and 7 we will analyze some examples by using the results obtained so far.

2. Preliminary results

Proof of Theorem 1.2. We first prove (1.10). Actually, we will prove two more refined results, which will be used in the proof of Theorems 1.3, 1.6, 1.9, 1.10.

Lemma 2.1. *Let q be any critical point for $\tilde{F}_m(\cdot;\Omega)$ in Ω and $g_q:\Omega\mapsto D$ any inverse Riemann map which satisfies $g_q(q)=0$. Define*

$$v_\varepsilon(w)=2\log\left(\frac{1+\varepsilon}{\varepsilon+|w|^2}\right),\quad w\in D,\tag{2.1}$$

and

$$u_\varepsilon(z)=v_\varepsilon\big(g_q(z)\big).$$

Then, as $\varepsilon\to 0^+$, we have

$$J_{8\pi}(u_\varepsilon)=1+\tilde{F}_m(q;\Omega)+\log\frac{|B_1|}{|\Omega|}+\frac{\varepsilon}{|c_0|^2}S(q)+O\big(\varepsilon^2\big).$$

Proof. Clearly, letting $f_q=g_q^{-1}$, we have

$$J_{8\pi}(u_\varepsilon)=-\frac{1}{16\pi}\int\limits_D|\nabla v_\varepsilon|^2+\log\frac{1}{|\Omega|}+\log\left(\int\limits_D\big|V\big(f_q(w)\big)\big|\big|f_q'(w)\big|^2e^{v_\varepsilon}\right).$$

It is easy to verify that

$$-\frac{1}{16\pi}\int\limits_D|\nabla v_\varepsilon|^2=\log\left(\frac{\varepsilon}{1+\varepsilon}\right)+\frac{1}{1+\varepsilon}.$$

On the other side, we have

$$\int\limits_D\big|V\big(f_q(w)\big)\big|\big|f_q'(w)\big|^2e^{v_\varepsilon}=I_1(\varepsilon,r_1)+I_2(\varepsilon,r_1),$$

where we define

$$I_1(\varepsilon,r_1)=\int\limits_{B_{r_1}}\big|V\big(f_q(w)\big)\big|\big|f_q'(w)\big|^2e^{v_\varepsilon}\,d\tau(w),$$

and

$$I_2(\varepsilon,r_1)=\int\limits_{D\setminus B_{r_1}}\big|V\big(f_q(w)\big)\big|\big|f_q'(w)\big|^2e^{v_\varepsilon}\,d\tau(w).$$

Clearly $e^{v_\varepsilon} \to \frac{1}{|w|^4}$ uniformly in $D \setminus B_{r_1}$, as $\varepsilon \to 0^+$, and it is not difficult to verify that

$$I_2(\varepsilon, r_1) = \int\limits_{D\setminus B_{r_1}} \frac{|\Phi(w) f_q'(w)|^2}{|w|^4}\, d\tau(w) + O(\varepsilon), \quad \text{as } \varepsilon \to 0^+.$$

Moreover, since $c_1 = 0$, we have

$$\begin{aligned}
I_1(\varepsilon, r_1) &= \int\limits_{D\setminus B_{r_1}} \left|\sum_{k=0}^{+\infty} c_k w\right|^2 e^{v_\varepsilon}\, d\tau(w) \\
&= 2\pi |c_0|^2 \int\limits_0^{r_1} e^{v_\varepsilon} \rho\, d\rho + 2\pi \sum_{k=2}^{+\infty} |c_k|^2 \int\limits_0^{r_1} \rho^{2k-3}\, d\rho + O(\varepsilon) \\
&= 2\pi |c_0|^2 \left(\frac{(1+\varepsilon)^2}{2\varepsilon} - \frac{(1+\varepsilon)^2}{2(\varepsilon + r_1^2)} \right) + 2\pi \sum_{k=2}^{+\infty} \frac{|c_k|^2}{2(k-1)} r_1^{2(k-1)} + O(\varepsilon) \\
&= 2\pi |c_0|^2 \frac{1}{2\varepsilon} - 2\pi |c_0|^2 \frac{1}{2r_1^2} + 2\pi \sum_{k=2}^{+\infty} \frac{|c_k|^2}{2(k-1)} r_1^{2(k-1)} + O(\varepsilon),
\end{aligned}$$

as $\varepsilon \to 0^+$. Thus, we may use the above expansions to conclude that

$$\begin{aligned}
J_{8\pi}(u_\varepsilon) &= 1 + \log\varepsilon + \log\frac{1}{|\Omega|} - \log\varepsilon + \log\Bigg[\pi |c_0|^2 + \varepsilon \Bigg(-2\pi |c_0|^2 \frac{1}{2r_1^2} + 2\pi \sum_{k=2}^{+\infty} \frac{|c_k|^2}{2(k-1)} r_1^{2(k-1)} \\
&\quad + \int\limits_{D\setminus B_{r_1}} \frac{|\Phi(w) f_q'(w)|^2}{|w|^4}\, d\tau(w) \Bigg) + O\big(\varepsilon^2\big)\Bigg] \\
&= 1 + \log\frac{\pi |c_0|^2}{|\Omega|} + \frac{\varepsilon}{\pi |c_0|^2} \Bigg(-2\pi |c_0|^2 \frac{1}{2r_1^2} + 2\pi \sum_{k=2}^{+\infty} \frac{|c_k|^2}{2(k-1)} r_1^{2(k-1)} \\
&\quad + \int\limits_{D\setminus B_{r_1}} \frac{|\Phi(w) f_q'(w)|^2}{|w|^4}\, d\tau(w) \Bigg) + O\big(\varepsilon^2\big) \\
&= 1 + \log|c_0|^2 + \log\frac{\pi}{|\Omega|} + \frac{1}{|c_0|^2} S(q)\varepsilon + O\big(\varepsilon^2\big) \\
&= 1 + \tilde{F}_m(q; \Omega) + \log\frac{|B_1|}{|\Omega|} + \frac{1}{|c_0|^2} S(q)\varepsilon + O\big(\varepsilon^2\big),
\end{aligned}$$

where we used the fact that

$$|c_0|^2 = \left|\Phi(0) f_q'(0)\right|^2 = \exp\big(F_m(0; 0, D)\big) \equiv \exp\big(\tilde{F}_m(0; D)\big) = \exp\big(\tilde{F}_m(q; \Omega)\big).$$

See also (3.5) below for more details concerning the last identity. □

Since the choice of the critical point q in Lemma 2.1 is arbitrary, it follows immediately that (1.10) holds.

To conclude the proof of Theorem 1.2, we will also need the following

Lemma 2.2. *$I_\lambda(\Omega)$ is continuous and increasing for $\lambda \in (0, 8\pi]$.*

Proof. Clearly $J_{\lambda_1}(u) < J_{\lambda_2}(u)$ for any $u \in H_0^1(\Omega)$ and $\lambda_1 < \lambda_2$, and then $I_{\lambda_1} \leqslant I_{\lambda_2}$. For any $\lambda_0 \in (0, 8\pi)$, and for any sequence $\lambda_n \to \lambda_0$, let $\{u_n\}$ be a sequence of maximizers, which of course exist because I_λ is attained in $(0, 8\pi)$. Then,

$$I_{\lambda_n} = J_{\lambda_n}(u_n) \geqslant J_{\lambda_n}(u), \quad \forall u \in H_0^1(\Omega). \tag{2.2}$$

Passing to the limit we easily conclude that $\liminf_{n\to+\infty} I_{\lambda_n} \geqslant J_{\lambda_0}(u)$, $\forall u \in H_0^1(\Omega)$, i.e. $\liminf_{n\to+\infty} I_{\lambda_n} \geqslant I_{\lambda_0}$. On the other side, since $\{u_n\}$ is bounded in $H_0^1(\Omega)$, we also conclude that

$$I_{\lambda_0} \geqslant J_{\lambda_0}(u_n) = J_{\lambda_0}(u_n) - J_{\lambda_n}(u_n) + J_{\lambda_n}(u_n) = o(1) + I_{\lambda_n},$$

as $n \to +\infty$, and we obtain $\limsup_{n\to+\infty} I_{\lambda_n} \leqslant I_{\lambda_0}$. If $\lambda_n \to 8\pi^-$, by using (2.2) we conclude once more that

$$\liminf_{n\to+\infty} I_{\lambda_n} \geqslant I_{8\pi},$$

and this time we may use the monotonicity of I_λ to obtain

$$\limsup_{n\to+\infty} I_{\lambda_n} \leqslant I_{8\pi}. \qquad \square$$

Observe that, for any $\rho \in \mathcal{P}(\Omega)$, where

$$\mathcal{P}(\Omega) = \left\{\rho \in L^1(\Omega) \;\middle|\; \rho \geqslant 0,\ \rho \log \rho \in L^1(\Omega),\ \|\rho\|_{L^1(\Omega)} = 1\right\}$$

the functional,

$$f_\lambda^*(\rho, V, \Omega) = \frac{\lambda}{2} \int\limits_\Omega \rho G_\Omega * \rho - \int\limits_\Omega \rho \log \rho + \int\limits_\Omega \rho \log \frac{V}{|\Omega|}, \tag{2.3}$$

is well defined, where G_Ω denotes the Green function defined in (1.2) and $G_\Omega * \rho$ the standard convolution. By arguing as [7], we also have the following

Lemma 2.3. *Let $\{u_k\}$ be a sequence of maximizers for J_{λ_k} as $\lambda_k \to 8\pi^-$. Assume that $\{u_k\}$ satisfies*

$$\lambda_k \frac{V e^{u_k}}{\int_\Omega V e^{u_k}} \rightharpoonup 8\pi \delta_q, \tag{2.4}$$

for some $q \in \Omega$. Then, the concentration point q is an absolute maximizer for $\tilde{F}_m(\cdot; \Omega)$ and

$$\limsup_{k\to+\infty} J_{\lambda_k}(u_k) \leqslant 1 + \tilde{F}_m(q; \Omega) + \log \frac{|B_1|}{|\Omega|}. \tag{2.5}$$

Proof. We may assume without loss of generality that $B_1(q) \Subset \Omega$. Putting

$$\rho_k = \frac{V e^{u_k}}{\int_\Omega V e^{u_k}}$$

and substituting in (2.3), we see that, for any k large enough, we have

$$\begin{aligned}
J_{\lambda_k}(u_k) &= f^*_{\lambda_k}(\rho_k, V, \Omega) \\
&= \frac{\lambda_k}{2} \int\limits_{B_1(q)} \rho_k G_{B_1(q)} * \rho_k - \int\limits_{B_1(q)} \rho_k \log \rho_k + o(1) + \frac{\lambda_k}{2} \int\limits_{B_1(q)} \rho_k \tilde{G}_\Omega * \rho_k \\
&\quad + \int\limits_\Omega \rho_k \log \frac{V}{|\Omega|} - \frac{\lambda_k}{2} \int\limits_{B_1(q)} \rho_k \tilde{G}_{B_1(q)} * \rho_k \\
&\leqslant \sup_{\rho \in \mathcal{P}(B_1)} f^*_{\lambda_k}(\rho, 1, B_1) + 4\pi \gamma_\Omega(q) + \log V(q) + \log \frac{|B_1|}{|\Omega|} + o(1) \\
&= \sup_{\rho \in \mathcal{P}(B_1)} f^*_{\lambda_k}(\rho, 1, B_1) + \tilde{F}_m(q; \Omega) + \log \frac{|B_1|}{|\Omega|} + o(1),
\end{aligned}$$

where we have used (2.4) and the fact that

$$\int\limits_{B_1(q)} \rho_k \tilde{G}_{B_1(q)} * \rho_k \to \tilde{G}_{B_1(q)}(q, q) = \tilde{G}_{B_1}(0, 0) = 0,$$

as $k \to +\infty$. It has been proved in [8] that, for any $\lambda \in (0, 8\pi)$,

$$\sup_{\rho \in \mathcal{P}(B_1)} f^*_\lambda(\rho, 1, B_1) = \sup_{v \in H^1_0(B_1)} \left\{ -\frac{1}{2\lambda} \int\limits_{B_1} |\nabla v|^2 + \log \fint\limits_{B_1} e^v \right\}.$$

Since the maximizers for this variational problem are well known and take up the simple radial expression (2.1), the same explicit evaluation of Lemma 2.1, shows that

$$\sup_{\rho \in P} f^*_{\lambda_k}(\rho_k, 1, B_1) \to 1 + 4\pi \gamma_{B_1}(0) = 1,$$

as $k \to +\infty$. Thus, (2.5) follows by using this result and the estimate above. In particular we can use Lemma 2.2 to conclude that

$$I_{8\pi} = \lim_{k \to +\infty} I_{\lambda_k} \leqslant \limsup_{k \to +\infty} J_{\lambda_k}(u_k) \leqslant 1 + \tilde{F}_m(q; \Omega) + \log \frac{|B_1|}{|\Omega|}. \tag{2.6}$$

At this point, by using (1.10), we immediately conclude that q is an absolute maximizer for $\tilde{F}(\cdot; \Omega)$. $\square$

We can finally conclude the proof of Theorem 1.2. By using Lemma 2.2, we see that $I_{8\pi} = \lim_{k \to +\infty} I_{\lambda_k}$, whenever $\lambda_k \to 8\pi^-$. Let $\{u_k\}$ be the corresponding sequence of maximizers. We argue by contradiction, and suppose that the strict inequality holds in (1.10), but $I_{8\pi}$ is not attained. If $\{u_k\}$ happens to be uniformly bounded, then a bootstrap argument shows that we may find a subsequence which converges in $H^1_0(\Omega)$ to a solution $u = u_{8\pi}$. As a consequence of Lemma 2.2, we see that $u_{8\pi}$ must be a maximizer for $J_{8\pi}$, so that $I_{8\pi}$ is attained. Thus we may assume without loss of generality that $\{u_k\}$ is unbounded. The Brezis–Merle theory [6] for Liouville type equations with singular data [5] then shows that there exists a subsequence which satisfies (2.4). Thus, we may apply Lemma 2.3 and conclude once more that (2.6) holds for some absolute maximizer q. It follows by (1.10) that indeed the equality sign holds in (1.10), which is the desired contradiction. □

Proof of Theorem 1.3. (i) ⇒ (ii) If the maximizers are uniformly bounded, and since solutions to (1.3) are unique for $\lambda \in (0, 8\pi)$, it is readily seen that they must converge, as $\lambda \to 8\pi^-$ to a solution $u_{8\pi}$ for (1.3) with $\lambda = 8\pi$. Lemma 2.2 implies that $u_{8\pi}$ is a maximizer for $J_{8\pi}$. Thus $I_{8\pi}$ is attained.

(ii) ⇒ (iii) If $I_{8\pi}$ is attained, clearly (1.3) admits a solution.

(iii) ⇒ (i) Assume by contradiction that the maximizers $\{u_\lambda\}$ where not uniformly bounded. Theorem 1.1 asserts that the first eigenvalue of the linearized problem for (1.3) at $u = u_{8\pi}$ is strictly positive. It then follows that the implicit function theorem can be applied and we may find a branch of uniformly bounded solutions $\{v_\lambda\}$ for any $\lambda - 8\pi < 0$ small enough. Thus, for any $\lambda - 8\pi < 0$ small enough, we conclude that $\|u_\lambda\|_\infty > \|v_\lambda\|_\infty$, which is a contradiction to the uniqueness of solutions for $\lambda \in (0, 8\pi)$. □

3. Proof of Theorems 1.5–1.6

Proof of Theorem 1.5. Let $f_q : D \mapsto \Omega$ be a Riemann map such that $f_q(0) = q$ and set $\tilde{u}_k(w) = u_k(f_q(w))$. Then $\{\tilde{u}_k\}$ satisfies

$$\begin{cases} -\Delta \tilde{u} = \lambda \dfrac{V(f_q(w))|f_q'(w)|^2 e^{\tilde{u}}}{\int_D V(f_q(w))|f_q'(w)|^2 e^{\tilde{u}}} & \text{in } D, \\ \tilde{u} = 0 & \text{on } \partial D. \end{cases} \tag{3.1}$$

Thus, (1.13) and (1.14) read

$$\lambda_k - 8\pi = V\big(f_q(0)\big)\big|f_q'(0)\big|^2 \tilde{\sigma}_k \Bigg(\int\limits_D \frac{\tilde{H}(w)}{|w|^4}\, d\tau(w) - \int\limits_{\mathbb{R}^2 \setminus D} \frac{1}{|w|^4}\, d\tau(w) + o(1) \Bigg),$$

$$\tilde{H}(w) = \frac{V(f_q(w))|f_q'(w)|^2}{V(f_q(0))|f_q'(0)|^2} - 1,$$

where $V(f_q(\cdot))$ satisfies (1.16). Clearly $\tilde{\sigma}_k = \sigma_k$. Thus,

$$\begin{aligned} \lambda_k - 8\pi &= \sigma_k \Bigg[\int\limits_D \frac{V(f_q(w))|f_q'(w)|^2 - V(f_q(0))|f_q'(0)|^2}{|w|^4}\, d\tau(w) \\ &\quad - \int\limits_{\mathbb{R}^2 \setminus D} \frac{V(f_q(0))|f_q'(0)|^2}{|w|^4}\, d\tau(w) + o(1) \Bigg] \\ &= \sigma_k \Bigg[\int\limits_D \frac{|\Phi(w) f_q'(w)|^2 - |\Phi(0) f_q'(0)|^2}{|w|^4}\, d\tau(w) - \big|\Phi(0) f_q'(0)\big|^2 \int\limits_{\mathbb{R}^2 \setminus D} \frac{1}{|w|^4}\, d\tau(w) \Bigg] \\ &= \sigma_k \Bigg[I_1(r_1) + I_2(r_1) + \pi \bigg(1 - \frac{1}{r_1^2}\bigg) |c_0|^2 - \pi |c_0|^2 \Bigg], \end{aligned}$$

where we define

$$I_1(r_1) = \int\limits_{B_{r_1}} \frac{|\Phi(w) f_q'(w)|^2 - |\Phi(0) f_q'(0)|^2}{|w|^4}\, d\tau(w),$$

and

$$I_2(r_1) = \int\limits_{D\setminus B_{r_1}} \frac{|\Phi(w) f_q'(w)|^2}{|w|^4}\, d\tau(w).$$

It is not difficult to verify that

$$I_1(r_1) = 2\pi \int\limits_0^{r_1} \sum_{k=2}^{+\infty} |c_k|^2 r^{2k-3}\, dr = 2\pi \sum_{k=2}^{+\infty} \frac{|c_k|^2}{2(k-1)} r_1^{2(k-1)}.$$

The conclusion follows collecting together the above expansions. $\square$

Proof of Theorem 1.6. Let q be a relative maximizer for $\tilde{F}_m(\cdot;\Omega)$, with $S(q) \leqslant 0$. We divide the proof in three steps.

Step 1. If $S(q) < 0$ and if q is a strict maximum for $\tilde{F}_m(\cdot;\Omega)$, then q is an absolute maximizer and the maximizers $\{u_\lambda\}$ for J_λ with $\lambda \in (0, 8\pi)$ satisfy (1.22) with concentration point q as $\lambda \to 8\pi^-$.

Since the maximum is strict, then it is non-degenerate according to the definitions in [14] and [15]. Then, there exists a sequence of 1-point blow-up solutions for (1.3), satisfying (1.22) with concentration point q. If $S(q) < 0$, Theorem 1.5 implies that $\lambda_k < 8\pi$ for any k large enough. By using Theorem 1.1, we conclude that $\{u_k\}$ coincides with a subset of the maximizer's set $\{u_\lambda\}$ for J_λ with $\lambda \in (0, 8\pi)$. Thus, the implications (i) $\Leftrightarrow$ (iii) of Theorem 1.3 and the Brezis–Merle theory readily imply that $\{u_\lambda\}$ must satisfy (1.22) at point q as $\lambda \to 8\pi^-$. Moreover q is an absolute maximizer by Lemma 2.3.

Step 2. If $S(q) \leqslant 0$, then q is an absolute maximizer of $\tilde{F}_m(\cdot;\Omega)$ and the maximizers $\{u_\lambda\}$ for J_λ with $\lambda \in (0, 8\pi)$ satisfy (1.22) with concentration point q as $\lambda \to 8\pi^-$.

We have the following

Lemma 3.1. *Let q be a relative maximum point for $\tilde{F}_m(\cdot;\Omega)$, with $S(q) \leqslant 0$. For any $\eta \in (0,1)$ define $f_q^{(\eta)}(\cdot) = f_q(\eta\cdot)$, $\Omega_\eta = f_q^{(\eta)}(D)$, and $\Phi^{(\eta)}(\cdot) = \Phi(\eta\cdot)$.*

For any $1-\eta > 0$ small enough, we have

$$q \in \Omega_\eta, \qquad \{p_1, \dots, p_m\} \subset \Omega_\eta, \qquad \left\{\frac{p_1^*}{\eta}, \dots, \frac{p_m^*}{\eta}\right\} \subset D, \tag{3.2}$$

and there exists a sequence of solutions $\{\tilde{u}_n^{(\eta)}\}$ for

$$\begin{cases} -\Delta \tilde{u} = \lambda \dfrac{|\Phi^{(\eta)}(w)(f_q^{(\eta)}(w))'|^2 e^{\tilde{u}}}{\int_\Omega |\Phi^{(\eta)}(w)(f_q^{(\eta)}(w))'|^2 e^{\tilde{u}}} & \text{in } D, \\ \tilde{u} = 0 & \text{on } \partial D, \end{cases} \tag{3.3}$$

which satisfies

$$\lambda_n \to 8\pi^-, \qquad \lambda_n \frac{|\Phi^{(\eta)}(w)(f_q^{(\eta)}(w))'|^2 e^{\tilde{u}_n}}{\int_\Omega |\Phi^{(\eta)}(w)(f_q^{(\eta)}(w))'|^2 e^{\tilde{u}_n}} \rightharpoonup 8\pi\delta_0. \tag{3.4}$$

Proof. Since $\Omega_\eta = f_q^{(\eta)}(D)$, $\Omega_\eta \to \Omega$ and $f_q^{(\eta)} \to f_q$, as $\eta \to 1^-$, it is clear that (3.2) holds for any $1-\eta$ small enough.

Clearly q is a relative maximum point for $\tilde{F}_m(\cdot;\Omega)$ if and only if $w=0$ is a relative maximum point for

$$\tilde{F}_m\big(f_q(w);\Omega\big) = \log\big[\big(1-|w|^2\big)^2 V\big(f_q(w)\big)\big|f_q'(w)\big|^2\big] \equiv \tilde{F}_m(w;D) = 4\pi\gamma_D(w) + \log\tilde{V}(w);$$
$$\tilde{V}(w) = V\big(f_q(w)\big)\big|f_q'(w)\big|^2 = \big|\Phi(w) f_q'(w)\big|^2. \tag{3.5}$$

Of course, this is equivalent to the fact that $w=0$ is a relative maximum point for $(1-|w|^2)^2|\Phi(w)f_q'(w)|^2$.

We are going to prove that $w=0$ is a strict maximum point for $\tilde{F}_m^{(\eta)}(w;D) = 4\pi\gamma_D(w) + \log|\Phi^{(\eta)}(w)(f_q^{(\eta)})'(w)|^2$.

Let $\{a_k\}$ and $\{b_k\}$ be the coefficients of the series expansions for f_q and Φ in D and B_{r_1} respectively. We can assume without loss of generality that

$$c_0 = b_0 a_1 = \Phi(0) f_q'(0) > 0.$$

Then, using the fact that $c_1 = 0$, it is easy to verify that

$$\big(1-|w|^2\big)^2\big|\Phi(w)f_q'(w)\big|^2 = |c_0|^2 + c_0 2\,\mathrm{Re}\big[c_2 w^2\big] - 2|c_0|^2|w|^2 + O\big(|w|^3\big)$$
$$= |c_0|^2 + 2c_0\langle \underline{w}, A\underline{w}\rangle + O\big(|\underline{w}|^3\big),$$

where, setting $w = w_1 + iw_2$, we denote by $\underline{w}$ the vector (w_1, w_2), by $\langle\cdot,\cdot\rangle$ the standard scalar product and, setting $c_2 = \xi_1 + i\xi_2$, we define $A = A(\{c_k\})$ to be the 2×2 matrix whose entries are

$$a_{11} = \xi_1 - c_0, \qquad a_{12} = -\xi_2 = a_{21}, \qquad a_{22} = -\xi_1 - c_0.$$

As far as we are concerned with the analysis of the definiteness of A, we can assume without loss of generality that $\xi_1 \geqslant 0$. Since $w=0$ is a maximum point, then, only one of the following situations may occur.

Either

(A1) $\det A > 0$ and $a_{11} < 0$, $a_{22} < 0$, or
(A2) $\det A = 0$, and $a_{11} < 0$, $a_{22} < 0$, or
(A3) $\det A = 0$, and at least one of a_{11} and a_{22} is zero.

Indeed, it is readily seen that any other case would imply that $w=0$ cannot be a local maximum point. Letting $\{c_k(\eta)\}$ be the coefficients of the power series relative to $|\Phi^{(\eta)}(w)(f_q^{(\eta)}(w))'|^2$, and since

$$c_k = \sum_{n=0}^{k}(n+1)a_{n+1}b_{k-n},$$

then we conclude that

$$c_k(\eta) = \eta^{k+1} c_k.$$

Thus, in particular $c_1(\eta) = 0$, because $c_1 = 0$. We are going to prove that for any $\eta \in (0, 1)$, then $A_\eta = A(\{c_k(\eta)\})$ is definite negative, i.e. that $w = 0$ is a strict maximum point for $\tilde{F}_m^{(\eta)}(w; D)$. It is enough to set up the worst case, that is (A3). In this case, observe that

$$\det A_\eta = \eta^2 c_0^2 - \eta^6\big(\xi_1^2 + \xi_2^2\big) > \eta^2\big[c_0^2 - \big(\xi_1^2 + \xi_2^2\big)\big] = 0.$$

Moreover, since $c_0 > 0$ and $\xi_1 \geqslant 0$, we can exclude the case where $c_0 = -\xi_1$. Then $c_0 = \xi_1$, and we conclude that $\xi_1 > 0$ and

$$-\eta^{2+1}\xi_1 - \eta c_0 < \eta^{2+1}\xi_1 - \eta c_0 = \eta\big(\eta^2\xi_1 - c_0\big) < \eta(\xi_1 - c_0) = 0.$$

Then, $\det A_\eta > 0$ and $a_{11}(\eta) < 0$, i.e. $w = 0$ is a strict maximum point for any $\eta \in (0, 1)$.

Thus, since $w = 0$ is a strict maximum point for $\tilde{F}_m^{(\eta)}(w; D)$, we can apply the results in [14] and [15] to conclude that for any such η, there exists a sequence of solutions for (3.3) which blows up as in (3.4).

We are left to prove that $\lambda_n \to 8\pi^-$.

First of all, we need to apply Theorem 1.4 to problem (3.3) on $\Omega = B_1$. To this end, we just need to verify that $|\Phi^{(\eta)}(w)(f_q^{(\eta)}(w))'|^2$ takes the form hV for some $\log h$ harmonic and continuous in $\overline{B_1}$ and V taking the form (1.4) for a suitable set of m point singularities in B_1. Indeed, by using (1.15), (1.19) and (3.2), we have

$$\big|\Phi^{(\eta)}(w)\big(f_q^{(\eta)}(w)\big)'\big|^2 = \left|\prod_{j=1,\dots,m}\left(\frac{p_j^* - \eta w}{1 - \eta\overline{p_j^*}w}\right)^{\alpha_j}\right|^2 \big|\big(f_q^{(\eta)}(w)\big)'\big|^2 \tag{3.6}$$

$$= \left|\prod_{j=1,\dots,m}\left(\frac{\frac{p_j^*}{\eta} - w}{1 - \frac{\overline{p_j^*}}{\eta}w}\right)^{\alpha_j}\right|^2 \left|\prod_{j=1,\dots,m}\left(\frac{\eta(1 - \frac{\overline{p_j^*}}{\eta}w)}{1 - \eta\overline{p_j^*}w}\right)^{\alpha_j}\right|^2 \big|\big(f_q^{(\eta)}(w)\big)'\big|^2$$

$$= \left[\exp\left(-4\pi \sum_{j=1,\dots,m} \alpha_j G_{B_1}\left(w, \frac{p_j^*}{\eta}\right)\right)\right] h(w) \tag{3.7}$$

where $\log(h(w)) = \log|(f_q^{(\eta)}(w))' \prod_{j=1,\dots,m}(\frac{\eta(1-\frac{\overline{p_j^*}}{\eta}w)}{1-\eta\overline{p_j^*}w})^{\alpha_j}|^2$ is easily seen to satisfy the required assumptions. Thus, we may apply Theorem 1.4 to problem (3.3) on $\Omega = B_1$. In particular, since the base domain is $B_1 \equiv D$ and the blow-up point is $w = 0$, by arguing as in the proof of Theorem 1.5, we verify that (1.23) holds true for $\lambda_n - 8\pi$, that is, $\lambda_n - 8\pi$ has the same sign of $S = S_\eta(0)$, where

$$S_\eta(0) = -\frac{\eta^2}{r_1^2}|c_0|^2 + \sum_{k=2}^{+\infty} \frac{\eta^{2(k+1)}|c_k|^2}{k-1} r_1^{2(k-1)} + \int_{D\setminus B_{\eta r_1}} \frac{|\Phi(\eta w) f'(\eta w)|^2}{|w|^4}\, d\tau(w)$$

$$= \eta^2\left[-\frac{1}{r_1^2}|c_0|^2 + \sum_{k=2}^{+\infty} \frac{\eta^{2k}|c_k|^2}{k-1} r_1^{2(k-1)} + \int_{D\setminus B_{\eta r_1}} \frac{|\Phi(\eta w) f'(\eta w)|^2}{|\eta w|^4}\, d\tau(\eta w)\right]$$

$$< \eta^2\left[-\frac{1}{r_1^2}|c_0|^2+\sum_{k=2}^{+\infty}\frac{|c_k|^2}{k-1}r_1^{2(k-1)}+\int\limits_{B_\eta\setminus B_{r_1}}\frac{|\Phi(w)f'(w)|^2}{|w|^4}\,d\tau(w)\right]<\eta^2 S(q)\leqslant 0.$$

Thus, $\lambda_n < 8\pi$ for any n large enough. $\square$

By using Lemma 3.1 we can conclude the proof of Step 2.

We argue by contradiction and suppose that there is no blowing up sequences of solutions for (1.3) as $\lambda \to 8\pi^-$ blowing up at q. The uniqueness result for (1.3), the Brezis–Merle theory and the implications (i) $\Leftrightarrow$ (ii) of Theorem 1.3 then show that the set of maximizers $\{u_\lambda\}$ for $\lambda \in (0, 8\pi)$, that is the unique solutions for (1.3), must satisfy either one of the following alternatives:

(B-1) $\|u_\lambda\|_\infty \leqslant \bar{C}$ for any $\lambda \in (0, 8\pi)$,
(B-2) $\{u_\lambda\}$ blows up at $q_0 \neq q$ as $\lambda \to 8\pi^-$.

We first discuss case (B-1).

If (B-1) holds true, then, in view of the implications (i) $\Leftrightarrow$ (iii) of Theorem 1.3, we may assume as well that (1.3) admits a solution $u_{8\pi}$ for $\lambda = 8\pi$. Set $C_{8\pi} = \|u_{8\pi}\|_\infty$. Observe that, for any $1-\eta$ small enough, and if $\{\tilde{u}_n^{(\eta)}\}$ denotes the sequence given by Lemma 3.1, then, using (3.6) and (3.7), we see that $u_n(z) = \tilde{u}_n^{(\eta)}(g_q^{(\eta)}(z))$ satisfies

$$\begin{cases} -\Delta u_n = \lambda_n \dfrac{h_\eta V_\eta e^{u_n}}{\int_\Omega h_\eta V_\eta e^{u_n}} & \text{in } \Omega_\eta, \\ u_n = 0 & \text{on } \partial\Omega_\eta, \end{cases} \tag{3.8}$$

where

$$V_\eta(z) = \exp\left(-4\pi \sum_{j=1,\dots,m} \alpha_j G_\eta(z, p_j)\right), \tag{3.9}$$

$$h_\eta(z) \to 1, \quad \text{uniformly as } \eta \to 1^-, \tag{3.10}$$

and $G_\eta(\cdot, p)$ denotes the Green function on Ω_η. Clearly $\log h_\eta$ is harmonic in Ω_η and continuous in $\overline{\Omega_\eta}$ uniformly with respect to η, as $\eta \to 1^-$. In particular, $\{u_n\}$ satisfies (1.22) on Ω_η with concentration point q as $\lambda_n \to 8\pi^-$ and $S_\eta(q) < 0$ (see the final part of the proof of Lemma 3.1). Thus, setting $\eta_k \to 1^-$ as $k \to +\infty$, and by arguing as in Step 1, we see that for any fixed $k \in \mathbb{N}$, the set of maximizers $\{u_k(\cdot, \lambda)\}$ for $\lambda \in (0, 8\pi)$, blows up as $\lambda \to 8\pi^-$ with concentration point q, and that q is an absolute maximizer for $\tilde{F}_m(\cdot; \Omega_{\eta_k})$. Observe that $u_k = u_k(z, \lambda)$ depends smoothly on λ. This is due to the fact that the first eigenvalue of the linearized operator for (3.8) is strictly positive, and the solution is unique, see Theorem 1.1. For any large constant $C > 2C_{8\pi}$, and for any $k \in \mathbb{N}$, there exists $\lambda_k \in (0, 8\pi)$ such that

$$\sup_\Omega u_k(z, \lambda_k) = C, \tag{3.11}$$

and

$$\sup_{B_{2\delta}(q)} u_k(z, \lambda_k) = C > 2 \sup_{\Omega\setminus B_\delta(q)} u_k(z, \lambda_k), \tag{3.12}$$

where δ is any small positive number such that $8\delta < \mathrm{dist}(q, p)$, for any other critical point p of $\tilde{F}_m(\cdot; \Omega)$. Thus, we may extract a subsequence of $u_k(\cdot, \lambda)$, which, in view of (3.9) and (3.10), converges uniformly to a solution $u(z, \lambda(C))$ of (1.3), where $\lambda(C) = \lim_{k\to+\infty} \lambda_k \leqslant 8\pi$. Since (1.3) admits a unique solution for $\lambda = 8\pi$, and since $C = \|u(\cdot, \lambda(C))\|_\infty > 2\|u_{8\pi}\|_\infty = 2C_{8\pi}$, then $\lambda(C) < 8\pi$ necessarily. Thus, putting $C = j$ and letting $j \to +\infty$, as a consequence of (3.12), we have a sequence of blow-up solutions for (1.3) with concentration point q, as $\lambda_j \to 8\pi^-$. This is the desired contradiction in case (B-1) holds true.

In case (B-2) holds true, we repeat this argument to obtain a sequence of solutions $u_k(\cdot, \lambda)$ for (3.8) which satisfies (3.11). Thus, we may extract a subsequence of $u_k(\cdot, \lambda)$, which converges uniformly to a solution $u(z, \lambda(C))$ of (1.3), where $\lambda(C) = \lim_{k\to+\infty} \lambda_k \leqslant 8\pi$. This time we observe that, since (B-2) holds true, then the implication (i) $\Leftrightarrow$ (iii) of Theorem 1.3 implies that (1.3) does not admit any solution for $\lambda = 8\pi$. Thus $\lambda(C) < 8\pi$ and we may argue as above to conclude that a sequence of blow-up solutions for (1.3) with concentration point q, as $\lambda_j \to 8\pi^-$ exists, in this case as well. This is the desired contradiction.

Thus, the conclusion of Step 2 follows by using Lemma 2.3 once more.

Step 3. By using Step 2, we conclude that if $S(q) \leqslant 0$, then q is an absolute maximizer of $\tilde{F}_m(\cdot; \Omega)$ and the maximizers $\{u_\lambda\}$ for J_λ with $\lambda \in (0, 8\pi)$ satisfy (1.22) with concentration point q as $\lambda \to 8\pi^-$. We argue by contradiction and assume that there exists another absolute maximizer for $\tilde{F}_m(\cdot; \Omega)$, say $q_0 \neq q$. If $S(q_0) \leqslant 0$, then we may apply Step 2 and come up with another sequence of solutions, say $\{u_k^{(0)}\}$ blowing up at q_0 as $\lambda_k \to 8\pi^-$. Since $\{u_{\lambda_k}\}$ and $\{u_k^{(0)}\}$ satisfy (1.22) with concentration points q and q_0 respectively, then we would have at least two distinct solutions for (1.3) for some $\lambda_k < 8\pi$, which is a contradiction to Theorem 1.1.

On the other side, if $S(q_0) > 0$, then Lemma 2.1 shows that $I_{8\pi} > 1 + \max_\Omega \tilde{F}_m(q; \Omega) + \log \frac{|B_1|}{|\Omega|}$, and then Theorem 1.2 implies that $I_{8\pi}$ is attained. Thus, the implications (i) $\Leftrightarrow$ (ii) of Theorem 1.3 imply that the maximizers $\{u_\lambda\}$ for J_λ with $\lambda \in (0, 8\pi)$ must be uniformly bounded, which is of course a contradiction. $\square$

4. Proofs of Theorems 1.9, 1.10 and of Corollary 1.8

Proof of Corollary 1.8. We argue by contradiction and assume that for some given $m \geqslant 1$ and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m)$ the set $\Omega_+^m(\underline{\alpha})$ is not open. Then, there exist $(p_1, \ldots, p_m)$ such that $I_{8\pi}$ is attained and a sequence $(p_1^{(n)}, \ldots, p_m^{(n)})$ such that $(p_1^{(n)}, \ldots, p_m^{(n)}) \to (p_1, \ldots, p_m)$ as $n \to +\infty$ and $I_{8\pi}^{(n)}$ is not attained for any $n \in \mathbb{N}$. Then, by using Theorem 1.6, we see that for any $n \in \mathbb{N}$ there exists only one absolute maximizer for $\tilde{F}_m^{(n)}(\cdot; \Omega)$, which we will denote by q_n. We may fix compact subsets $K \subset K_1 \Subset \{\Omega \setminus \{p_1, \ldots, p_m\}\}$ with the property that $\{q_n\} \subset K$ and $\{p_1^{(n)}, \ldots, p_m^{(n)}\} \subset \{\Omega \setminus K_1\}$ for any $n \in \mathbb{N}$. Clearly $\tilde{F}_m^{(n)}(z; \Omega)$ depends smoothly on $(p_1^{(n)}, \ldots, p_m^{(n)})$ for $z \in K$. In particular q_n is continuous as a function of $(p_1^{(n)}, \ldots, p_m^{(n)})$. Since, by using Theorem 1.6, we have $S(q_n) \leqslant 0$, and since S is continuous as a function of q, we conclude that $S(q) \leqslant 0$, for some absolute maximizer satisfying $q_n \to q$. Thus Theorem 1.6 implies that q is unique and $I_{8\pi}$ is not attained, which is of course a contradiction. $\square$

Proof of Theorem 1.9. It follows by Lemma 2.1 that if $S(q) > 0$ for some absolute maximizer of $\tilde{F}_m(\cdot; \Omega)$, then $I_{8\pi} > 1 + \max_{\overline{\Omega}} \tilde{F}_m(\cdot; \Omega) + \log \frac{|B_1|}{|\Omega|}$. It then follows by Theorem 1.2 that $I_{8\pi}$ is attained.

To prove the opposite implication, we argue by contradiction and suppose that $I_{8\pi}$ is attained and $S(q) \leqslant 0$ for all absolute maximums of $\tilde{F}_m(\cdot; \Omega)$. Then, by using Theorem 1.6, we see that there exists only one absolute maximizer, say q_0, and the maximizers $\{u_\lambda\}$, $\lambda \in (0, 8\pi)$, blow up at q_0. It follows by the implications (i) $\Leftrightarrow$ (ii) of Theorem 1.3 that $I_{8\pi}$ cannot be attained. $\square$

Proof of Theorem 1.10. If $I_{8\pi}(\Omega) > 1 + \max_{\overline{\Omega}} \tilde{F}_m(\cdot; \Omega) + \log \frac{|B_1|}{|\Omega|}$, then $I_{8\pi}$ is attained by Theorem 1.2.

On the other side, Theorem 1.9 asserts that (ii) of Theorem 1.3 (that is, $I_{8\pi}$ is attained) is equivalent to $S(q) > 0$ for an absolute maximizer q of F_m. It follows immediately by Lemma 2.1 that $I_{8\pi}(\Omega) > 1 + \max_{\overline{\Omega}} \tilde{F}_m(\cdot; \Omega) + \log \frac{|B_1|}{|\Omega|}$. $\square$

5. The case $\Omega = B_1$ and $m = 1$

In this section we discuss the case where $\Omega = B_1$ and $m = 1$ and set $\alpha = \alpha_1$ and $p = p_1$.

The case $p = 0$ has been already discussed in [4], where it has been proved by other methods that solutions for (1.3) blowing up as $\lambda \to 8\pi$, must obey to $\lambda > 8\pi$ for any $\lambda - 8\pi$ small enough. This is of course what we expect, since (1.3) admits in this case an explicit branch of radial solutions for any $\lambda \in (0, 8\pi(1+\alpha))$ for any $\alpha > 0$, and then Theorem 1.3 implies indeed that $\lambda > 8\pi$ for one-point blow-up solutions.

In particular we have a third proof of this fact in this particular case. It is provided by Corollary 1.11, since it is trivial to verify that $\tilde{F}_1$ admits a full inner circle as set of absolute maximizers.

Thus, let us restrict our attention to the case where $p \neq 0$. We have the following

Theorem 5.1. *For any $p \in B_1 \setminus \{0\}$, there exist $0 < \alpha_- \leqslant \alpha_+ < +\infty$ such that if $\alpha \leqslant \alpha_-$ then $I_{8\pi}$ is not attained while if $\alpha > \alpha_+$ then $I_{8\pi}$ is attained.*

Proof. By using the rotation invariance of the problem, let us assume that

$$p = \xi, \quad \xi \in (0,1).$$

A straightforward evaluation shows that, for any $\alpha > 0$, the critical points $q = q_x + iq_y$ of $\tilde{F}_1$, which in this case reads (see (1.9))

$$\tilde{F}_1(z; B_1) = 2\log\left(1 - |z|^2\right) + 2\alpha \log\left|\frac{z-\xi}{1-z\xi}\right|,$$

are solutions of

$$q_y = 0, \qquad P(q_x) = 0,$$

where $P(q_x) = 2\xi q_x^3 - (2(1+\xi^2) + \alpha(1-\xi^2))q_x^2 + 2\xi q_x + \alpha(1-\xi^2)$. Since $P(1) = -2(1-\xi)^2$, then P clearly admits at least one zero in $(1, +\infty)$, and since $\tilde{F}_1(x; B_1)$ has at least one absolute maximum in *each* interval $(-1, \xi)$ and $(\xi, 1)$, we conclude that $\tilde{F}_1(z; B_1)$ admits only two critical points in B_1, that is

$$q_- = q_-(\xi, \alpha) \in (-1, \xi), \qquad q_+ = q_+(\xi, \alpha) \in (\xi, 1).$$

We observe however that $P(-1) = -2(1+\xi)^2$, $P(0) = \alpha(1-\xi^2)$, $P(\xi) = \alpha(1-\xi^2)^2 < P(0)$ and $P(1) = -2(1-\xi)^2$. Since P is of degree 3 and admits already one root in $(\xi, 1)$ and one root in $(1, +\infty)$, we conclude that

$$q_- \in (-1, 0).$$

At this point we may check that for any $\alpha > 0$,

$$\tilde{F}_1(q_-) > \tilde{F}_1(q_+), \quad \forall \xi \in (0,1).$$

Using

$$\phi_\xi(z) = \frac{z-\xi}{1-z\xi}, \quad z \in B_1,$$

we see that ϕ_ξ^{-1} maps the family of disks of radius $\{B_\rho\}_{\rho \in (0,1)}$ conformally onto a family of balls $\{B_R(\bar{z})\}$, where $R = R(\xi, \rho) \in (0,1)$ is strictly increasing from 0^+ to 1^- and $\bar{z} = \bar{z}(\xi, \rho)$, $\bar{z}(\xi, \rho) \in$

$(0,\xi)$, is strictly decreasing from ξ^+ to 0^+ as ρ increases from 0^+ to 1^-. In particular, $B_{R_1}(\bar{z}_1) \subset B_{R_2}(\bar{z}_2)$ whenever $R_i = R(\xi,\rho_i)$ and $\bar{z}_i = \bar{z}(\xi,\rho_i)$ with $\rho_2 > \rho_1$. Let $R_\pm$ and $\bar{z}_\pm$ satisfy $q_\pm \in \partial B_{R_\pm}(\bar{z}_\pm)$. Clearly $|\phi_\xi(q_-)| > |\phi_\xi(q_+)|$, if and only if $B_{R_+}(\bar{z}_+) \subset B_{R_-}(\bar{z}_-)$. We observe at this point that for any small α, $\tilde{F}_1$ is close to $\tilde{F}_0(z;B_1) := 4\pi\gamma(z)$ in C^2 norm on any compact subset of $B_1 \setminus \{\xi\}$. In particular, since $P(0) \to 0$ and $P(\xi) \to 0$, as $\alpha \to 0^+$, we conclude that $q_- \to 0^-$ and $q_+ \to \xi^+$ as $\alpha \to 0^+$. Indeed, by using the implicit function theorem, it is easy to prove that q_- is decreasing and q_+ is increasing with α. For later use, we remark that we may also conclude easily that $q_- \to (-1)^+$, as $\alpha \to +\infty$. Thus, we may choose α small enough to guarantee that $B_{R_+}(\bar{z}_+) \subset B_{R_-}(\bar{z}_-)$ and $|q_-| < |q_+|$, so that, in particular,

$$\tilde{F}_1(q_-) = 2\log\bigl(1-|q_-|^2\bigr) + 2\alpha\log\bigl|\phi_\xi(q_-)\bigr| > \tilde{F}_1(q_+).$$

Then q_- is the unique absolute maximizer for $\tilde{F}_1$ and α small enough. Let us apply Theorem 1.9 and work directly in the base domain by using (1.23),

$$S(q) = -\frac{1}{r_1^2}|c_0|^2 + \sum_{k=2}^{+\infty}\frac{|c_k|^2}{k-1}r_1^{2(k-1)} + \frac{1}{\pi}\int\limits_{D\setminus B_{r_1}}\frac{|\Phi(w)f'(w)|^2}{|w|^4}\,d\tau(w),$$

where

$$f(w) = \frac{q_- - w}{1-q_- w}, \qquad f'(w) = \frac{|q_-|^2-1}{(1-q_- w)^2}, \qquad \Phi(w) = \left(\frac{p^*-w}{1-p^* w}\right)^{\alpha},$$
$$p^* = \frac{q_- - \xi}{1-q_-\xi}, \qquad r_1 = \bigl|p^*\bigr|.$$

Observe that, as $\alpha \to 0^+$, we have

$$q_- \to 0^-, \qquad p^* \to (-\xi)^-, \qquad r_1 \to \xi,$$

and

$$f'(w) \to -1, \qquad \Phi(w) \to 1, \quad \text{uniformly in } D\setminus B_r, \quad \text{for any } r > \xi.$$

Then, we have

$$\frac{1}{\pi}\int\limits_{D\setminus B_{r_1}}\frac{|\Phi(w)f'(w)|^2}{|w|^4}\,d\tau(w) \to \frac{1}{\xi^2} - 1, \quad \text{as } \alpha \to 0^+.$$

Moreover, since the $\{c_k\}$ are nothing but the coefficients of the Cauchy product series for $\Phi f'$ in B_{r_1}, it is readily seen that

$$-\frac{1}{r_1^2}|c_0|^2 + \sum_{k=2}^{+\infty}\frac{|c_k|^2}{k-1}r_1^{2(k-1)} \to -\frac{1}{\xi^2}, \quad \text{as } \alpha \to 0^+,$$

and then,

$$S\bigl(q_-(\xi,\alpha)\bigr) \to -1, \quad \text{as } \alpha \to 0^+.$$

We conclude that, for any fixed $\xi \in (0,1)$, there exists $\alpha_- = \alpha_-(\xi)$ such that for any $\alpha \leqslant \alpha_-$, $I_{8\pi}$ is not attained.

Next, we will prove that there exists $\alpha_+ = \alpha_+(\xi)$ such that for any $\alpha > \alpha_+$, $I_{8\pi}$ is attained. To obtain this result, we will use Theorem 1.10. Let us define

$$\tilde{I}(\alpha) = I_{8\pi}(B_1) - \Big[1 + \max_{B_1} \tilde{F}_1(\cdot; B_1)\Big].$$

Clearly we have

$$\begin{aligned}\tilde{I}(\alpha) &> J_{8\pi}(0) - \Big[1 + \max_{B_1} \tilde{F}_1(\cdot; B_1)\Big] \\ &\geqslant \log\biggl(\fint_{B_1} \bigl|\phi_\xi(z)\bigr|^{2\alpha}\, d\tau(z)\biggr) - 1 - 2\log\bigl(1 - |q_-|^2\bigr) - 2\alpha \log\bigl|\phi_\xi(q_-)\bigr|.\end{aligned}$$

Observe that, since $q_- \in \partial B_{R_-}(\bar{z}_-)$, $\bar{z}_- \in (0,\xi)$ and $q_- \in (-1,0)$, then, *for any* α, the sector

$$C(q_-) = \left\{ |z| \in \bigl(|q_-|, 1\bigr),\ \arg z \in \left(\frac{3}{4}\pi, \frac{5}{4}\pi\right)\right\}$$

satisfies $C(q_-) \subset B_1 \setminus B_{R_-}(\bar{z}_-)$. On the other side, by the definition of $B_{R_-}(\bar{z}_-)$, we conclude that $|\phi_\xi(z)| > |\phi_\xi(q_-)|$ for any $z \in B_1 \setminus B_{R_-}(\bar{z}_-)$. Thus,

$$\log\biggl(\fint_{B_1} \bigl|\phi_\xi(z)\bigr|^{2\alpha}\, d\tau(z)\biggr) - 2\alpha \log\bigl|\phi_\xi(q_-)\bigr| > -\log|B_1| + \log\biggl(\frac{\pi}{4}\bigl(1 - |q_-|^2\bigr)\biggr),$$

and in particular,

$$\tilde{I}(\alpha) > -1 - \log 4 - \log\bigl(1 - |q_-|^2\bigr) > 0,$$

whenever $|q_-| > 1 - \frac{1}{4e}$. Since, as already remarked above, $q_- \to (-1)^+$ monotonically as α increases from 0^+ to $+\infty$, we conclude that there exists $\alpha_+(\xi)$ such that for any $\alpha > \alpha_+(\xi)$, $\tilde{I}(\alpha) > 0$. Then, the definition of $\tilde{I}(\alpha)$ and Theorem 1.10 together imply that (1.3) admits a solution for $\lambda = 8\pi$ for any such α. □

6. The case where Ω is symmetric and $m = 1$

We consider the case where Ω is symmetric with respect to the x_1 and x_2 axes with $m = 1$ one singularity $p = p_1 \in \Omega$, $\alpha = \alpha_1$. Of course, since Ω is simply connected, we have $0 \in \Omega$. Then we have

Theorem 6.1. *If $p = 0$, then $I_{8\pi}$ is attained for any $\alpha > 0$.*

Remark. In particular it is not true that for α small $I_{8\pi}$ is attained if and only if it is attained for the problem with $\alpha = 0$. Indeed, in case $\Omega = B_1$ with no singularities, it is well known that $I_{8\pi}$ is not attained. We also point out that, in view of Remark 1, this result holds true for example on any regular N-polygon with $N \in \mathbb{N}$ even.

Proof of Theorem 6.1. By using the symmetry with respect to the x_1 axis, see for example [1, p. 232], we see that if a Riemann map is normalized in such a way that $g(q)=0$ and $\arg[g'(q)]>0$, for some *real* $q\in\Omega$, we have $g(\overline{w})=\overline{g}(w)$. Thus, according to our notations, let us set $g=g_q$. In view of the symmetry with respect to the x_2 axis, by taking $q=0$ and by using this result after a rotation, we conclude that g_q is symmetric with respect to the x_1 and x_2 axes. In particular, it follows that $\gamma_\Omega(z)$ and $G_\Omega(z,0)$ are symmetric with respect to the x_1 and x_2 axes. At this point, let us set $p=0$, so that

$$\tilde{F}_1(z;\Omega)=4\pi\gamma_\Omega(z)-4\pi\alpha G_\Omega(z,0).$$

Clearly $\tilde{F}_1$ is symmetric with respect to the x_1 and x_2 axes. Since $\tilde{F}_1(z;\Omega)\to-\infty$ as $z\to 0$ and $z\to\partial\Omega$, it admits at least one interior absolute maximizer, which we denote by $q_\alpha\in\Omega\setminus\{0\}$. Clearly, if either $q_\alpha\in\{x_1=0\}$ or $q_\alpha\in\{x_2=0\}$ then $\tilde{F}_1$ admits at least two absolute maximizers. If $q_\alpha\notin\{x_1=0\}\cup\{x_2=0\}$ then $\tilde{F}_1$ admits at least four absolute maximizers. In any case, as a consequence of Corollary 1.11, $I_{8\pi}$ is attained for any $\alpha>0$. □

7. The case where $\Omega=B_1$ and $m=2$

We consider the case $\Omega=B_1$ with $m=2$ symmetric singularities $p=p_1=-p_2$ and $\alpha=\alpha_1=\alpha_2$. In this situation we obtain

Theorem 7.1. *For any $p\in B_1\setminus\{0\}$, there exists $\overline{\alpha}>0$ such that $I_{8\pi}$ is attained for any $\alpha>\overline{\alpha}$.*

Proof. Without loss of generality let $p_1=(0,\xi)$ for some $\xi\in(0,1)$. Then

$$\tilde{F}_2(z;B_1)=2\log\bigl(1-|z|^2\bigr)+2\alpha\log\left|\frac{z-\xi}{1-z\xi}\right|+2\alpha\log\left|\frac{z+\xi}{1+z\xi}\right|.$$

Clearly $\tilde{F}_2$ is symmetric with respect to the x_1 and x_2 axes. As a consequence, by arguing as in Section 6, we may use Corollary 1.11 and conclude that $I_{8\pi}$ is attained whenever $z=0$ does not happen to be the unique absolute maximizer for $\tilde{F}_2$. Thus, our next aim will be to prove that there exists $\overline{\alpha}=\overline{\alpha}(\xi)>0$ such that

$$\tilde{F}_2(0;B_1)<\tilde{F}_2\bigl(i\xi^2;B_1\bigr), \tag{7.1}$$

for any $\alpha>\overline{\alpha}$. Indeed, observe that (7.1) is equivalent to

$$2\log|\xi|^{2\alpha}<2\log\left[\bigl(1-|\xi|^2\bigr)\left(\frac{2\xi^2}{1+\xi^4}\right)^\alpha\right],$$

that is

$$\left(\frac{1+\xi^4}{2}\right)^\alpha<\bigl(1-|\xi|^2\bigr). \tag{7.2}$$

Elementary considerations show that there exists a unique $\bar{\xi}=\bar{\xi}(\alpha)\in(0,1)$ such that

$$\left(\frac{1+\bar{\xi}^4}{2}\right)^\alpha=\bigl(1-|\bar{\xi}|^2\bigr),$$

and that (7.2) is satisfied for any $\xi<\bar{\xi}$. In particular we can verify immediately that $\bar{\xi}(\alpha)$ is increasing. Since $(\frac{1+\xi^4}{2})^\alpha\to 0$ pointwise for $\xi\in[0,1)$ as $\alpha\to+\infty$, then $\bar{\xi}\to 1^-$ as $\alpha\to+\infty$. Thus, for any

$\xi \in (0,1)$ there exists $\bar{\alpha} = \bar{\alpha}(\xi) > 0$ such that (7.1) is satisfied for any $\alpha > \bar{\alpha}$ and we conclude that $z = 0$ cannot be a maximizer for $\tilde{F}_2$. In particular $\tilde{F}_2$ admits at least two maximizers and then $I_{8\pi}$ is attained for any $\alpha > \bar{\alpha}$. $\square$

We conclude this series of examples with a result concerning the case where Ω is any bounded and simply connected domain with a finite number of conical type points (see [9] for more details), with $m \geqslant 1$ singularities $\{p_1, \dots, p_m\}$ and $\underline{\alpha} = (\alpha_1, \dots, \alpha_m)$. Let Γ denote the set of absolute maximizers for γ_Ω and assume that $\{p_1, \dots, p_m\} \cap \Gamma = \emptyset$. For any $q \in \Gamma$, let $S_0(q)$ be the quantity (1.20) corresponding to the "regular" case $\underline{\alpha} = \underline{0}$. We refer to [9] for more details concerning this point.

Theorem 7.2. *Suppose that $S_0(q) \neq 0$ for at least one $q \in \Gamma$. Then, for any $|\underline{\alpha}|$ small enough, $I_{8\pi}$ is attained if and only if it is attained for $\underline{\alpha} = \underline{0}$.*

Proof. We first claim that there exists a compact subset $K \Subset \{\Omega \setminus \{p_1, \dots, p_m\}\}$ with the property that any absolute maximizer for $\tilde{F}_m(z;\Omega)$, which we denote by $q_{\underline{\alpha}}$, satisfies $\{q_{\underline{\alpha}}\} \subset K$ as $|\underline{\alpha}| \to 0$.

We argue by contradiction and suppose that there exists a sequence $\underline{\alpha}_n \to \underline{0}$, such that there exists a sequence of maximizers $\{q_{\underline{\alpha}_n}\}$ for $\tilde{F}_m^{(n)}(z;\Omega)$ such that $q_{\underline{\alpha}_n}$ converges to one of the p_j's as $|\underline{\alpha}_n| \to 0$. We may assume without loss of generality that $q_{\underline{\alpha}_n} \to p_1$ as $|\underline{\alpha}_n| \to 0$. Let $V_n = V(\underline{\alpha}_n, z)$ be the weight defined by (1.4). Since $V_n \leqslant 1$ for any $z \in \Omega$, and since $q_{\underline{\alpha}_n}$ is a maximizer, for any $n \in \mathbb{N}$ we have

$$4\pi\gamma_\Omega(z) + \log\bigl[V_n(z)\bigr] = \tilde{F}_m^{(n)}(z;\Omega) \leqslant \tilde{F}_m^{(n)}(q_{\underline{\alpha}_n};\Omega) \leqslant 4\pi\gamma_\Omega(q_{\underline{\alpha}_n}), \quad \forall z \in \Omega.$$

In particular we conclude that for any $n \in \mathbb{N}$ and for any fixed $q \in \Gamma$, we have

$$4\pi\gamma_\Omega(q) + \log\bigl[V_n(q)\bigr] \leqslant 4\pi\gamma_\Omega(q_{\underline{\alpha}_n}).$$

As $n \to +\infty$ we conclude that

$$4\pi\gamma_\Omega(q) \leqslant 4\pi\gamma_\Omega(p_1),$$

which is the desired contradiction.

At this point we observe that the same argument shows that there exists at least one maximizer $q_{\underline{\alpha}}$ such that $q_{\underline{\alpha}} \to q \in \Gamma$ as $|\underline{\alpha}| \to 0$. Then we see that $S(q_{\underline{\alpha}}) \to S_0(q)$ because S is continuous as a function of $q_{\underline{\alpha}}$ as far as $\{q_{\underline{\alpha}}\} \subset K$. We claim that $S_0(q) \neq 0$. Indeed, if $S_0(q) = 0$, then a result in [9] asserts that q is the unique maximizer for γ_Ω. This is a contradiction to our assumption that $S_0(\bar{q}) \neq 0$ for at least one absolute maximizer of γ_Ω. Thus $S_0(q) \neq 0$, and we conclude that for any small $|\underline{\alpha}|$, $I_{8\pi}$ is attained if and only if it is attained for $\underline{\alpha} = \underline{0}$. This is because S_0 plays the same role (see [9]) for the "regular" problem that S plays in the singular one. $\square$

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